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On some classes of operators by Fujii and Nakamoto related to p-hyponormal and paranormal operators

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Abstract

Recently, we introduced class A as a new class of operators in [18]. Class A is defined by an operator inequality, and also the definition of class A is similar to that of paranormality defined by a norm inequality. We showed that every log-hyponormal operator belongs to class A and every class A operator is paranormal in [18]. As generalizations of class A and paranormality, class $\mathrm{A}(p,r)$ was introduced in [11] and absolute-(p,r)-paranormality was introduced in [30]. Moreover, Fujii-Nakamoto [12] introduced class $\mathrm{F}(p,r,q)$ and (p,r,q)-paranormality which are further generalizations of these classes.

In this report, we shall show some inclusion relations among the families of class $\mathrm{F}(p,r,q)$ and (p,r,q)-paranormality, and we shall show the result on powers of class $\mathrm{F}(p,r,q)$ operators.

1 Introduction

In this report, a capital letter means a bounded linear operator on a complex Hilbert space H. An operator T is said to be positive (denoted by $T \geq 0$) if $(Tx,x) \geq 0$ for all $x \in H$, and also an operator T is said to be strictly positive (denoted by $T > 0$) if T is positive and invertible.

As extensions of hyponormal operators, i.e., $T^*T \geq TT^*$, p-hyponormal operators for $p > 0$ defined by $(T^*T)^p \geq (TT^*)^p$ and log-hyponormal operators defined by $\log T^*T \geq \log TT^*$ for an invertible operator T are well known. And also an operator T is p-quasihyponormal for $p > 0$ if T is p-hyponormal on $\overline{R(T)}$. It is easily obtained that every p-hyponormal operator is q-hyponormal for $p > q > 0$ by Löwner-Heinz theorem "*$A \geq B \geq 0$ ensures $A^\alpha \geq B^\alpha$ for any $\alpha \in [0,1]$*," and every invertible p-hyponormal operator for $p > 0$ is log-hyponormal since $\log t$ is an operator monotone function. We remark that log-hyponormality is sometimes regarded as 0-hyponormality since $\frac{X^p - I}{p} \to \log X$ as $p \to +0$ for $X > 0$.

An operator T is paranormal if $\|T^2x\| \geq \|Tx\|^2$ for every unit vector $x \in H$. It has been studied by many authors, so there are too many to cite their references, for instance, [3][13][17] and [21]. Ando [3] showed that *every p-hyponormal operator for $p > 0$ and log-hyponormal operator is paranormal.*

Recently, in [18], we introduced class A defined by $|T^2| \geq |T|^2$ where $|T| = (T^*T)^{\frac{1}{2}}$, and we showed that every log-hyponormal operator belongs to class A and every class A operator is paranormal. It turns out that these results contain another proof of Ando's result stated above. We remark that class A is defined by an operator inequality and paranormality is defined by a norm inequality, and their definitions appear to be similar forms.

And also we introduced two families of classes of operators based on class A and paranormality in [18] as follows: An operator T belongs to class A(k) for $k > 0$ if $(T^*|T|^{2k}T)^{\frac{1}{k+1}} \geq |T|^2$, and also an operator T is absolute-k-paranormal for $k > 0$ if $\||T|^kTx\| \geq \|Tx\|^{k+1}$ for every unit vector $x \in H$. Particularly an operator T is a class A (resp. paranormal) operator if and only if T is a class A(1) (resp. absolute-1-paranormal) operator. It was shown in [18] that the classes of invertible class A(k) operators and absolute-k-paranormal operators constitute parallel and increasing lines, that is, invertible class A(k) $\subseteq$ invertible class A(l) and absolute-k-paranormal $\subseteq$ absolute-l-paranormal for $0 < k \leq l$.

On the other hand, Fujii-Izumino-Nakamoto [7] introduced p-paranormality for $p > 0$ defined by $\||T|^pU|T|^px\| \geq \||T|^px\|^2$ for every unit vector $x \in H$, where $T = U|T|$ is the polar decomposition of T. We remark that 1-paranormality equals paranormality. As generalizations of class A(k), absolute-k-paranormality and p-paranormality, Fujii-Jung-S.H.Lee-M.Y.Lee-Nakamoto [11] introduced class A(p,r) and Yamazaki-Yanagida [30] introduced absolute-(p,r)-paranormality as follows:

Definition.

(1) *For each $p > 0$ and $r > 0$, an operator T belongs to class $A(p,r)$ if*

$$(|T^*|^r|T|^{2p}|T^*|^r)^{\frac{r}{p+r}} \geq |T^*|^{2r},$$

and let class $AI(p,r)$ be the class of all invertible class $A(p,r)$ operators.

(2) *For each $p > 0$ and $r > 0$, an operator T is absolute-(p,r)-paranormal if*

$$\||T|^p|T^*|^rx\|^r \geq \||T^*|^rx\|^{p+r} \tag{1.1}$$

for every unit vector $x \in H$.

It was pointed out that class A$(k,1)$ equals class A(k) in [28]. And also, in [30], it was shown that absolute-$(k,1)$-paranormality equals absolute-k-paranormality and absolute-(p,p)-paranormality equals p-paranormality. Moreover class AI$(\frac{1}{2},\frac{1}{2})$ equals the class of invertible and w-hyponormal operators ($|\tilde{T}| \geq |T| \geq |(\tilde{T})^*|$ where $T = U|T|$ is the polar decomposition of T and $\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$) introduced by Aluthge-Wang [2]. We should remark that the families of class AI(p, r) determined by operator inequalities and absolute-(p, r)-paranormality determined by norm inequalities constitute two increasing lines on $p > 0$ and $r > 0$ whose origin is log-hyponormality (see section 2).

Moreover, as a continuation of the discussion in [11], Fujii-Nakamoto [12] introduced the following classes of operators.

Definition.

(1) *For each $p > 0$, $r \geq 0$ and $q \geq 1$, an operator T belongs to class F(p, r, q) if*

$$(|T^*|^r|T|^{2p}|T^*|^r)^{\frac{1}{q}} \geq |T^*|^{\frac{2(p+r)}{q}}. \tag{1.2}$$

(2) *For each $p > 0$, $r \geq 0$ and $q > 0$, an operator T is (p, r, q)-paranormal if*

$$\left\| |T|^pU|T|^r x \right\|^{\frac{1}{q}} \geq \left\| |T|^{\frac{p+r}{q}} x \right\| \tag{1.3}$$

for every unit vector $x \in H$, where $T = U|T|$ is the polar decomposition of T.

We remark that class F$(p, r, \frac{p+r}{r})$ equals class A(p, r), and we obtain that $(p, r, \frac{p+r}{r})$-paranormality equals absolute-(p, r)-paranormality in the next section. Thus many re-seachers have been discussed parallel families of classes of operators which are generalizations of class A and paranormality.

In this report, firstly, we obtain more precise inclusion relations among the families of class F(p, r, q) and (p, r, q)-paranormality from the view of monotonicity of class A(p, r) and absolute-(p, r)-paranormality. Secondly, we give a characterization of log-hyponormal operators via class F(p, r, q) and (p, r, q)-paranormality. Lastly, we obtain the result on powers of class F(p, r, q) operators.

2 Background and preliminaries

Firstly, we obtain another expression of (p, r, q)-paranormality without using U which appears in the polar decomposition of T, and it causes that $(p, r, \frac{p+r}{r})$-paranormality equals absolute-(p, r)-paranormality.

Proposition 1. *For each $p > 0$, $r > 0$ and $q \geq 1$, an operator T is (p, r, q)-paranormal if and only if*

$$\left\| |T|^p|T^*|^r x \right\|^{\frac{1}{q}} \geq \left\| |T^*|^{\frac{p+r}{q}} x \right\| \tag{2.1}$$

for every unit vector $x \in H$.

Corollary 2. *For each $p > 0$ and $r > 0$, $(p, r, \frac{p+r}{r})$-paranormality equals absolute-(p, r)-paranormality.*

Next, to explain the background of the classes of operators discussed in this paper, we have to state the following celebrated order preserving operator inequality.

Theorem F (Furuta inequality [14]).

If $A \geq B \geq 0$, then for each $r \geq 0$,

(i) $$(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{1}{q}} \geq (B^{\frac{r}{2}}B^pB^{\frac{r}{2}})^{\frac{1}{q}}$$

and

(ii) $$(A^{\frac{r}{2}}A^pA^{\frac{r}{2}})^{\frac{1}{q}} \geq (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{1}{q}}$$

hold for $p \geq 0$ and $q \geq 1$ with $(1+r)q \geq p+r$.

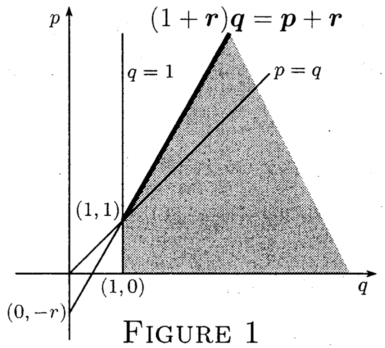

FIGURE 1

We remark that Theorem F yields Löwner-Heinz theorem when we put $r = 0$ in (i) or (ii) stated above. Alternative proofs of Theorem F were given in [5] and [24] and also an elementary one page proof in [15]. It was shown in [25] that the domain drawn for p, q and r in the Figure 1 is the best possible one for Theorem F.

Fujii-Nakamoto [12] observed that class F(p, r, q) derives from Theorem F and (p, r, q)-paranormality corresponds to class F(p, r, q), and also they showed the following Theorem A.1.

Theorem A.1 ([12]).

(i) *For a fixed $k > 0$, T is k-hyponormal if and only if T belongs to class $F(2kp, 2kr, q)$ for all $p > 0$, $r \geq 0$ and $q \geq 1$ with $(1+2r)q \geq 2(p+r)$, i.e., T belongs to class $F(p, r, q)$ for all $p > 0$, $r \geq 0$ and $q \geq 1$ with $(k+r)q \geq p+r$.*

(ii) *If T belongs to class $F(p_0, r_0, q_0)$ for $p_0 > 0$, $r_0 \geq 0$ and $q_0 \geq 1$, then T belongs to class $F(p_0, r, q_0)$ for any $r \geq r_0$.*

(iii) *If T belongs to class $F(p_0, r_0, q_0)$ for $p_0 > 0$, $r_0 \geq 0$ and $q_0 \geq 1$, then T belongs to class $F(p_0, r_0, q)$ for any $q \geq q_0$.*

(iv) *If T belongs to class $F(p, r, q)$ for $p > 0$, $r \geq 0$ and $q \geq 1$, then T is (p, r, q)-paranormal.*

(v) *If T is (p_0, r_0, q_0)-paranormal for $p_0 > 0$, $r_0 \geq 0$ and $q_0 > 0$, then T is (p_0, r_0, q)-paranormal for any $q \geq q_0$.*

(vi) *If* T *is* $(p_0, r_0, 1)$*-paranormal for* $p_0 > 0$ *and* $r_0 \geq 0$*, then* T *is* $(p_0, r, 1)$*-paranormal for any* $r \geq r_0$.

(vii) *If* T *is* $(p, r, 1)$*-paranormal for* $p > 0$ *and* $r \geq 0$*, then* T *is* $\max\{p, r\}$*-paranormal.*

On the other hand, chaotic order is defined by $\log A \geq \log B$ for positive and invertible operators A and B. Chaotic order is weaker than usual order $A \geq B$ since $\log t$ is an operator monotone function. As a characterization of chaotic order, the following Theorem B.1 was obtained by using Theorem F.

Theorem B.1 ([6][8][16][26]). *Let* A *and* B *be positive invertible operators. Then the following properties are mutually equivalent:*

(i) $\log A \geq \log B$.

(ii) $(B^{\frac{p}{2}} A^p B^{\frac{p}{2}})^{\frac{1}{2}} \geq B^p$ *for all* $p \geq 0$.

(iii) $(B^{\frac{r}{2}} A^p B^{\frac{r}{2}})^{\frac{r}{p+r}} \geq B^r$ *for all* $p \geq 0$ *and* $r \geq 0$.

We remark that the equivalence between (i) and (ii) was shown in [4].

Noting that class $\mathrm{F}(p, r, \frac{p+r}{r})$ equals class $\mathrm{A}(p, r)$, we can verify that class $\mathrm{A}(p, r)$ derives from Theorem B.1. On class $\mathrm{A}(p, r)$ and absolute-(p, r)-paranormality, the following Theorem A.2 and Theorem A.3 were shown in [11] and [30], respectively. We remark that Figure 2 expresses the inclusion relations shown in Theorem A.2 and Theorem A.3.

Theorem A.2 ([11]).

(i) T *is log-hyponormal if and only if* T *belongs to class* $AI(p, r)$ *for all* $p > 0$ *and* $r > 0$.

(ii) *If* T *belongs to class* $AI(p_0, r_0)$ *for* $p_0 > 0$ *and* $r_0 > 0$*, then* T *belongs to class* $AI(p, r)$ *for any* $p \geq p_0$ *and* $r \geq r_0$.

(iii) *If* T *belongs to class* $A(p_0, r_0)$ *for* $p_0 > 0$ *and* $r_0 > 0$*, then* T *belongs to class* $A(p_0, r)$ *for any* $r \geq r_0$.

Theorem A.3 ([30]).

(i) T *is log-hyponormal if and only if* T *is invertible and absolute-*(p, r)*-paranormal for all* $p > 0$ *and* $r > 0$.

(ii) *If T is absolute-(p_0, r_0)-paranormal for $p_0 > 0$ and $r_0 > 0$, then T is absolute-(p, r)-paranormal for any $p \geq p_0$ and $r \geq r_0$.*

(iii) *If T belongs to class $A(p, r)$ for $p > 0$ and $r > 0$, then T is absolute-(p, r)-paranormal.*

(iv) *If T is absolute-(p, r)-paranormal for $p > 0$ and $r > 0$, then T is normaloid, i.e., $\|T\| = r(T)$ where $r(T)$ is the spectral radius of T.*

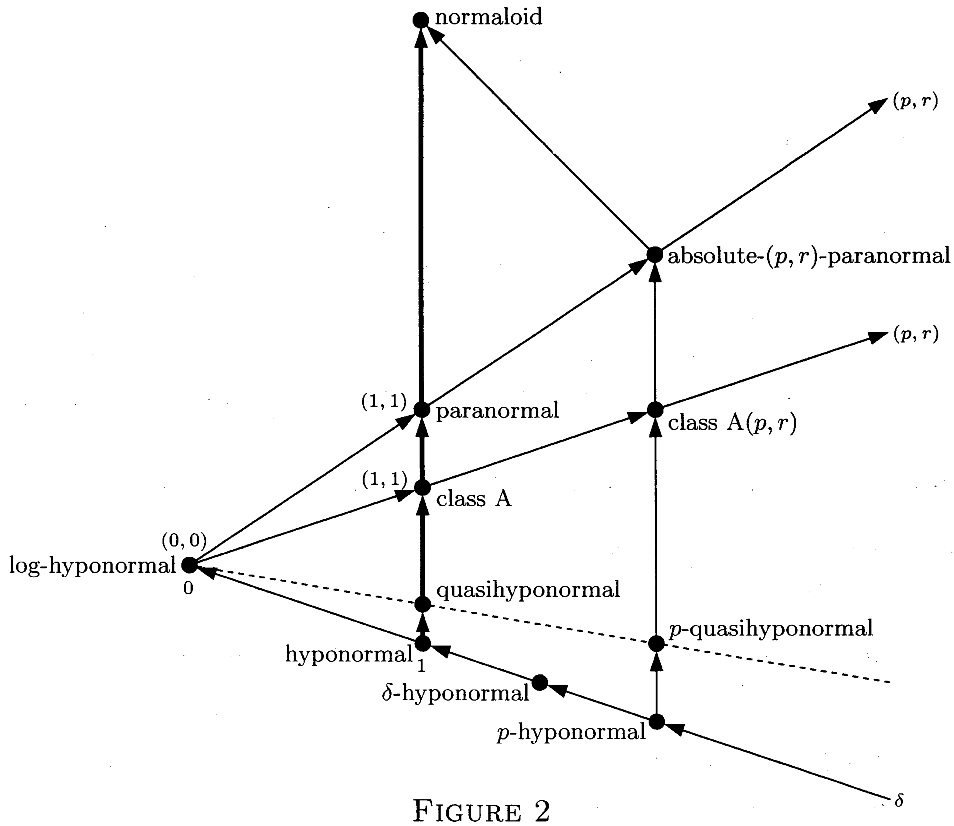

FIGURE 2

Theorem A.2 and Theorem A.3 state that the families of class AI(p, r) determined by operator inequalities and absolute-(p, r)-paranormality determined by norm inequalities have monotonicity on $p > 0$ and $r > 0$, and log-hyponormality regarded as class AI$(0, 0)$ or absolute-$(0, 0)$-paranormality, namely they constitute two increasing lines whose origin is log-hyponormality.

3 Inclusion relations

In this section, we discuss monotonicity of class F(p, r, q) and (p, r, q)-paranormality.

In section 2, we verified that class $\mathrm{A}(p,r)$ derives from Theorem B.1, and also we explained that Theorem A.2 and Theorem A.3 state that the families of class $\mathrm{AI}(p,r)$ and absolute-(p,r)-paranormality constitute two increasing lines on $p>0$ and $r>0$ whose origin is log-hyponormality.

On the other hand, as a parallel result to Theorem B.1, Theorem F also leads to the following Theorem B.2.

Theorem B.2 ([9][10]). *For positive operators A and B, $A^{\delta}\geq B^{\delta}$ for a fixed $\delta>0$ if and only if*

$$(B^{\frac{r}{2}}A^{p}B^{\frac{r}{2}})^{\frac{\delta+r}{p+r}}\geq B^{\delta+r}$$

holds for all $p\geq\delta$ and $r\geq 0$.

Considering these matters, it seems natural that we rewrite class $\mathrm{F}(p,r,q)$ and (p,r,q)-paranormality by class $\mathrm{F}(p,r,\frac{p+r}{\delta+r})$ and $(p,r,\frac{p+r}{\delta+r})$-paranormality when we discuss monotonicity of class $\mathrm{F}(p,r,q)$ and (p,r,q)-paranormality on p and r. In fact, we obtain the following results on monotonicity of class $\mathrm{F}(p,r,\frac{p+r}{\delta+r})$ and $(p,r,\frac{p+r}{\delta+r})$-paranormality. And also the following Figure 3 represents the inclusion relations shown in this section.

Proposition 3. *The following assertions hold for each $p>0$ and $r>0$:*

(i) *T is p-quasihyponormal if and only if T belongs to class $F(p,r,1)$ if and only if T is $(p,r,1)$-paranormal.*

(ii) *T is p-quasihyponormal if and only if T is $(p,0,1)$-paranormal.*

Theorem 4. *Let T be a class $F(p_0,r_0,\frac{p_0+r_0}{\delta+r_0})$ operator for $p_0>0$, $r_0\geq 0$ and $-r_0<\delta\leq p_0$. Then the following assertions hold:*

(i) *T belongs to class $F(p_0,r,\frac{p_0+r}{\delta+r})$ for any $r\geq r_0$.*

(ii) *If T is invertible and $0\leq\delta\leq p_0$, then T belongs to class $F(p,r,\frac{p+r}{\delta+r})$ for any $p\geq p_0$ and $r\geq r_0$.*

Theorem 5. *Let T be a $(p_0,r_0,\frac{p_0+r_0}{\delta+r_0})$-paranormal operator for $p_0>0$, $r_0\geq 0$ and $\delta>-r_0$. Then the following assertions hold:*

(i) *If $-r_0<\delta\leq p_0$, then T is $(p_0,r,\frac{p_0+r}{\delta+r})$-paranormal for any $r\geq r_0$.*

(ii) *If $0\leq\delta$, then T is $(p,r_0,\frac{p+r_0}{\delta+r_0})$-paranormal for any $p\geq p_0$.*

(iii) *If $0\leq\delta\leq p_0$, then T is $(p,r,\frac{p+r}{\delta+r})$-paranormal for any $p\geq p_0$ and $r\geq r_0$.*

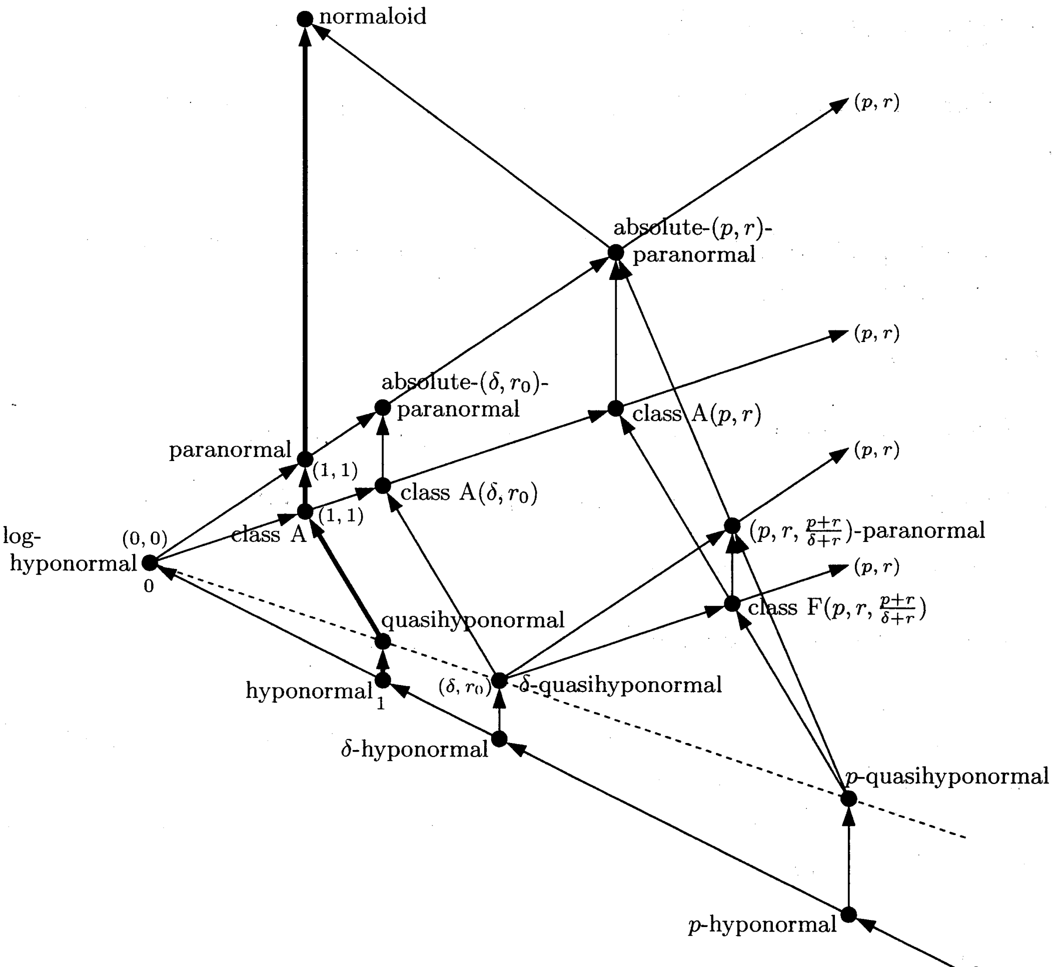

FIGURE 3

Proposition 3, Theorem 4 and Theorem 5 assert that invertible class $\mathrm{F}(p,r,\frac{p+r}{\delta+r})$ and $(p,r,\frac{p+r}{\delta+r})$-paranormality for $\delta>0$ constitute two increasing lines for $p\geq\delta>0$ and $r\geq r_0>0$ which have δ-quasihyponormality as the origin since δ-quasihyponormality equals class $\mathrm{F}(\delta,r_0,1)$ or $(\delta,r_0,1)$-paranormality. And also, in case $\delta=0$, (i) and (ii) of Theorem 4 means (iii) and (ii) of Theorem A.2, respectively, and Theorem 5 means (ii) of Theorem A.3. Therefore monotonicity of invertible class $\mathrm{F}(p,r,\frac{p+r}{\delta+r})$ and $(p,r,\frac{p+r}{\delta+r})$-paranormality for $\delta>0$ is parallel to monotonicity of class $\mathrm{AI}(p,r)$ and absolute-(p,r)-paranormality since invertible δ-quasihyponormality (i.e., δ-hyponormality) approaches log-hyponormality as $\delta\to+0$.

Remark. We remark that Proposition 1 does not hold for $r=0$ and $q=1$ since (2.1) holds for $p>0$, $r=0$ and $q=1$, i.e., $\big\||T|^p x\big\|\geq\big\||T^*|^p x\big\|$ for every unit vector $x\in H$ if and only if T is p-hyponormal, but T is $(p,0,1)$-paranormal for $p>0$ if and only if T is p-quasihyponormal by (ii) of Proposition 3.

4 Log-hyponormality

As a characterization of log-hyponormal operators, the following Theorem D.1 was obtained.

Theorem D.1 ([11][29][30]). *Let T be an invertible operator. Then the following assertions are mutually equivalent:*

(i) *T is log-hyponormal.*

(ii) *T belongs to class $A(p,p)$, i.e., class $AI(p,p)$ for all $p > 0$.*

(iii) *T belongs to class $A(p,r)$, i.e., class $AI(p,r)$ for all $p > 0$ and $r > 0$.*

(iv) *T is p-paranormal for all $p > 0$.*

(v) *T is absolute-(p,r)-paranormal for all $p > 0$ and $r > 0$.*

(i)$\Leftrightarrow$(ii)$\Leftrightarrow$(iii) was obtained in [11], and also (i)$\Leftrightarrow$(iv) and (i)$\Leftrightarrow$(iv)$\Leftrightarrow$(v) were obtained in [29] and [30], respectively.

As an extension of Theorem D.1 via class $\mathrm{F}(p,r,q)$ and (p,r,q)-paranormality, we have the following Theorem 6.

Theorem 6. *Let T be an invertible operator. Then the following assertions are mutually equivalent for any fixed $\alpha \in (0,1]$:*

(i) *T is log-hyponormal.*

(ii) *T belongs to class $F(p,p,\frac{2}{\alpha})$ for all $p > 0$.*

(iii) *T belongs to class $F(p,r,\frac{p+r}{r\alpha})$ for all $p > 0$ and $r > 0$.*

(iv) *T is $(p,p,\frac{2}{\alpha})$-paranormal for all $p > 0$.*

(v) *T is $(p,r,\frac{p+r}{r\alpha})$-paranormal for all $p > 0$ and $r > 0$.*

We remark that Theorem 6 ensures Theorem D.1 by putting $\alpha = 1$.

5 Powers of class $\mathbf{F}(p, r, q)$ operators

On powers of p-hyponormal and log-hyponormal operators, Aluthge-Wang [1] and Yamazaki [27] showed the following results (see also [19][20][23]).

Theorem E.1 ([1]). *Let T be a p-hyponormal operator for $0 < p \leq 1$. Then T^n is $\frac{p}{n}$-hyponormal for all positive integer n.*

Theorem E.2 ([27]). *Let T be a log-hyponormal operator. Then T^n is also log-hyponormal for all positive integer n.*

On the other hand, on powers of class $\mathrm{A}(p, r)$ operators, Yamazaki [28] showed the following Theorem E.3 (see also [22]).

Theorem E.3 ([28]). *Let T be a class $AI(p, r)$ operator for $0 < p \leq 1$ and $0 < r \leq 1$. Then T^n belongs to class $AI(\frac{p}{n}, \frac{r}{n})$ for all positive integer n.*

In this section, we obtain the following result on powers of class $\mathrm{F}(p, r, q)$ operators.

Theorem 7. *Let T be an invertible class $F(p, r, q)$ operator for $0 < p \leq 1$, $0 \leq r \leq 1$ and $q \geq 1$ with $rq \leq p + r$. Then T^n belongs to class $F(\frac{p}{n}, \frac{r}{n}, q)$ for all positive integer n.*

Theorem 7 interpolates Theorem E.1 and Theorem E.3 in case T is invertible. In fact, Theorem 7 yields Theorem E.1 by putting $q = 1$ and $r = 0$, and also Theorem 7 yields Theorem E.3 by putting $q = \frac{p+r}{r}$.

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